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An Iterative Numerical Algorithm for Solving Multi-Parameter Inverse Problems of Evolutional Partial Differential Equations

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The iterative numerical algorithm of the pulse-spectrum technique (PST) is extended and developed to solve the multi-parameter inverse problems of one-dimensional evolutional partial differential equations (wave equations or diffusion equations). It has the practical advantages of universality, economy of programing, economy of data acquisition, and economy of computing costs. Without the real measurement data, numerical simulations are carried out only for the two-parameter inverse problems of a one-dimensional linear wave equation to test the feasibility and to study the general characteristics of the PST. It is found that the PST does give excellent results and it is as robust as in the single parameter case.

INTRODUCTION

The multi-parameter inverse problem of an evolutional partial differential equation (a wave equation or a diffusion equation) is to determine several unknown parameters or coefficients of the governing partial differential equation simultaneously as functions of space variables from information of the solution of the partial differential equation on a portion of the boundary surface or the interior. Its applications can be found in many branches of science and engineering, e.g., to infer the structure of Earth from seismic wave measurements in geophysical prospecting, to determine the nature of the irregularities of solid materials from wave measurements in nondestructive evaluation, to acquire the ocean structure from acoustic wave measurements in physical oceanography, to infer the transmissivity and the permeability from know data of the pressure and pressure gradient in oil reservoir and aquifer simulations, respectively, to determine the thermal properties of a nonhomogeneous solid from surface temperature measurements, etc. Since the solution of the multi-parameter inverse problem provides one with much more infor-

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mation on the corresponding physical problem than the solution of the single parameter inverse problem, the development of an efficient and versatile numerical method for solving the multi-parameter inverse problems of the evolutional partial differential equations becomes very important.

Up to now, there have been many efforts made by scientists, engineers, and mathematicians to develop methods for solving the multi-parameter inverse problems of the wave equations. For example, Nigul [1] and Nigul and Engelbrecht [2] have presented a perturbation method for solving a two-parameter inverse problem of a one-dimensional nonlinear wave equation, but it is limited to the layered media and lacks of generality. Raz [3] has developed a direct Born inversion technique for solving the two-parameter inverse problem of a three-dimensional linear wave quation; however, its requirement of the constant reference velocity and the lack of a systematic procedure to improve the accuracy of the approximate solutions post severe restrictions on the range of applicability of this method. Rose and Opsal [4] have gone a step further by using Born approximation to solve a three-parameter inverse problem of a system of three-dimensional linear wave equations; again it not only suffers the same deficiencies as the Born inversion technique of Raz but also requires a large amount of measurement data at different spatial locations to provide sufficient resolution to this method. Based on an inverse scattering theory, Coen [5] has presented a method for solving a two-parameter inverse problem of a two-dimensional linear wave equation; its lack of generality and the requirement of the precise knowledge of either the mximum value or the minimum value of the unknown parameters make this method impractical. On the other hand, there is very little or no efforts in the development of methods for solving the multi-parameter inverse problems of the diffusion equations.

In this paper, the pulse-spectrum technique (PST) is extended for solving the multi-parameter inverse problems of the evolutional partial differential equations. The basic isea of the PST is that data are measured in the time-domain with compact support and the synthesis of the unknown parameters is carried out numerically in the complex frequency-domain by an iterative algorithm where each cycle of iteration consists of solving an elliptic boundary value problem (Laplace transformed initial boundary value problem) several times and a Fredholm integral equation of the first kind once. The pulse-spectrum technique was first introduced by Tsien and Chen [6] for solving a single parameter one-dimensional inverse problem in fluid dynamics. Then it was further developed to have the capability of handling the noisy, poor distributed, and inadequately measured data by Chen and Tsien [7]. Later, the PST was used to solve single parameter inverse problems in one-dimensional electromagnetic wave propagation by Tsien and Chen [8], in synthesis of nonuniform transmission lines by Chen and Weng [9] and in one-dimensional dynamical structural identification by Chen and Lin [10]. For solving inverse problems of nonlinear partial differential equations, it was extended successfully to solve a single parameter inverse problem of a one-dimensional nonlinear wave equation by Hatcher and Chen [11]. The PST was also modified to a solve single parameter inverse problem of a one-dimensional linear diffusion equation by Chen and Liu [12]. Moreover, for

solving inverse problems of the evoluational partial differential equations in arbitrary multi-dimensional domain, the PST has been used to solve single parameter inverse problems of a two-dimensional linear wave equation by Chen and Liu [13] and a two-dimensional linear diffusion equation by Liu and Chen [14]. Finally, the discretized version of the iterative algorithm of the PST under idealized conditions was proved to converge quadratically by Chen [15] and the proof of the uniqueness of the solution in general will be given by Xie and Chen [16].

It has been shown in the above-mentioned literature that the PST fares very well in regard to the following four practical criteria for the evaluation of any numerical method for solving inverse problems.

(a) Universality criterion. Whether or not a numerical method which is effective in one space-dimensional and single-parameter inverse problems can be extended with similar success into higher space-dimensional and multi-parameter inverse problems? Whether or not a solution method which is effective for solving inverse problems of either hyperbolic type of partial differential equations or parabolic type of partial differential equations can be extended to solve the inverse problems of the other type of partial differential equations with similar success and minimum efforts?

(b) Economy of programing effort criterion. The numerical method should be as close to the nondedicated program as possible, because the existing practices of programing new dedicated numerical methods for every special type of problems can be extremely expensive in many practical circumstances. Furthermore, the computer code (program) should also contain (as many as possible) the modules where the canned subroutines can be readily called upon.

(c) Economy of data acquisition criterion. The numerical method should be able to keep the difficulties and the cost expenditure of acquiring or measuring the necessary data ffor a successful calculation to minimum.

(d) Economy of computing cost criterion. The numerical method should keep the cost of I0 and CPU times and memory storage to a minimum.

The PST fares very well in regard to the universality criterion. This is because the finite difference method or the finite element method is just as adaptable to solve any higher dimensional elliptic boundary value problem with arbitrary finite domain as to solve a one-dimensional boundary value problem and the Tikhonov's regularization method is also as adaptable to solve any higher dimensional Fredholm integral equation of the first kind as to solve the one-dimensional case. Since the synthesis procedure of the PST is carried out in the frequency-domain where the elliptic partial differential equations obtained by Laplace transforming the wave equation or the diffusion equation are very similar, the PST can be used to solve inverse problems of both types of the evolutional equations with trivial changes in the computer code of PST.

The programing for the PST is basically nondedicated, because, as mentioned above, the change-over from solving the inverse problems of a class of linear hyper-

bolic partial differential equations to solving the inverse problems of a class of parabolic partial differential equations is simply a matter of changing a few instructions in the general elliptic solver. Furthermore, one does not have to program a subroutine for the elliptic equation solver, for there are a abundance of the difference and finite element computer codes for solving general elliptic boundary value problems available in the public domain. Hence the PST fares very well in regard to the economy of programing effort criterion.

As it has been demonstrated in [13, 14], the measurement data are needed only at a small portion of the boundary surface or in a small region of the interior for the PST to solve inverse problems successfully. Hence the PST again fares rather well in regard to the economy of data acquisition criterion in comparison with other methods.

Finally, the PST seems also to fare rather well in regard to the economy of computing cost criterion. However, the actual computing costs depend very much on the particular computer hardwares and softwares, and one cannot be sure of this until a bench mark comparison test is performed. Nevertheless, a new automatic adaptive-grid method will be developed for the PST so that maximum accuracy is achieved with a minimum number of spatial grid points.

In this paper, the capability of the PST for solving multi-parameter inverse problems of the evolutional partial differential equations is demonstrated. For simplicity, the formulation of the extended PST is presented only for the inverse problems of a one-dimensional evolutional partial differential equation. In particular, numerical simulations are carried out for two-parameter inverse problems to test the feasibility and to study the intrinsic characteristics of the PST without the real measurement data. Finally, a comprehensive discussion of the numerical results and their implication in actually implementing the PST are given.

Numerical Algorithm (Pulse-Spectrum Technique)

Consider the following initial-boundary value problem of a one-dimensional linear evolution equation,

$$\partial\{k(x)\,\partial u/\partial x\}/\partial x + \beta(x)u - \alpha(x)\,\partial u/\partial t - \rho(x)\,\partial^2 u/\partial t^2 = 0, \qquad 0 < x < 1,\ 0 < t < \infty,$$
$$u(x,0) = \partial u(x,0)/\partial t = 0, \qquad u(0,t) = e(t), \qquad \text{and} \qquad u(1,t) = f(t), \tag{1}$$

where $k(x)$, $\alpha(x)$, and $\rho(x)$ are positive functions; $k(x)$ is continuous, $\alpha(x)$, $\beta(x)$, and $\rho(x)$ are piecewise continuous.

Here the inverse problem is to determine the unknown coefficients $k(x)$, $\alpha(x)$, $\beta(x)$, and $\rho(x)$ from the known initial conditions, the known boundary conditions $e(t)$ and $f(t)$, and the additionally measured auxiliary data,

$$\partial u(0,t)/\partial x = h_1(t), \qquad \partial u(1,t)/\partial x = h_2(t),$$
$$u(x_1,t) = h_3(t) \qquad \text{and} \qquad u(x_2,t) = h_4(t), \quad 0 < x_1 \neq x_2 < 1. \tag{2}$$

Assuming that the initial-boundary data and the auxiliary data are all Laplace transformable, the PST calls for the transformation of (1) and (2) so that the entire system is transformed from the time-domain to the complex frequency-domain, and the corresponding transformed system is

$$\begin{gathered}\partial\{k(x)\,\partial v/\partial x\}/\partial x+\beta(x)v-s\alpha(x)v-s^2\rho(x)v=0, \qquad 0<x<1,\\ v(0,s)=E(s) \qquad \text{and} \qquad v(1,s)=F(s),\end{gathered} \tag{3}$$

and

$$\begin{gathered}\partial v(0,s)/\partial x=H_1(s), \qquad \partial v(1,s)/\partial x=H_2(s),\\ v(x_1,s)=H_3(s) \qquad \text{and} \qquad v(x_2,s)=H_4(s), \quad 0<x_1\neq x_2<1,\end{gathered} \tag{4}$$

where $v(x,s)$, $E(s)$, $F(s)$, $H_1(s)$, $H_2(s)$, $H_3(s)$, and $H_4(s)$ are the Laplace transformations of $u(x,t)$, $e(t)$, $f(t)$, $h_1(t)$, $h_2(t)$, $h_3(t)$, and $h_4(t)$, respectively. Now, the inverse problem is to determine $k(x)$, $\alpha(x)$, $\beta(x)$, and $\rho(x)$ from $E(s)$, $F(s)$, $H_1(s)$, $H_2(s)$, $H_3(s)$, and $H_4(s)$.

The iterative numerical algorithm begins by setting

$$\begin{aligned}v_{n+1}&=v_n+\delta v_n, \qquad & k_{n+1}&=k_n+\delta k_n,\\ \alpha_{n+1}&=\alpha_n+\delta\alpha_n, \qquad & \beta_{n+1}&=\beta_n+\delta\beta_n, \qquad n=0,1,2,3,\ldots,\end{aligned} \tag{5}$$

and

$$\rho_{n+1}=\rho_n+\delta\rho_n,$$

where $k_0(x)$, $\alpha_0(x)$, $\beta_0(x)$, and $\rho_0(x)$ are the initial guesses for the unknown coefficients $k(x)$, $\alpha(x)$, $\beta(x)$, and $\rho(x)$, respectively, and $\|\delta v_n\|<\|v_n\|$, $\|\delta k_n\|<\|k_n\|$, $\|\delta\alpha_n\|<\|\alpha_n\|$, $\|\delta\beta_n\|<\|\beta_n\|$, and $\|\delta\rho_n\|<\|\rho_n\|$. Since the unknown coefficients can be measured directly at the two end points, without loss of generality one can assume that these coefficients are known only at the two end points. These conditions are not crucial to the success of PST, but they do make the computation easier.

Upon substituting (5) into (3) and neglecting terms of order δ^2 and higher, one obtains a system for v_n,

$$\begin{gathered}\partial(k_n\,\partial v_n/\partial x)/\partial x+\beta_n(x)v_n-s\alpha_n(x)v_n-s^2\rho_n(x)v_n=0, \qquad 0<x<1,\\ v_n(0,s)=E(s) \qquad \text{and} \qquad v_n(1,s)=F(s),\end{gathered} \tag{6}$$

and a system for δv_n,

$$\begin{aligned}&\partial(k_n\,\partial\delta v_n/\partial x)/\partial x+\beta_n(x)\,\delta v_n-s\alpha_n(x)\,\delta v_n-s^2\rho_n(x)\,\delta v_n\\ &\quad=s^2\delta\rho_n(x)v_n+s\delta\alpha_n(x)v_n-\delta\beta_n(x)v_n-\partial(\delta k_n\,\partial v_n/\partial x)/\partial x, \qquad 0<x<1,\\ &\qquad\delta v_n(0,s)=\delta v_n(1,s)=0.\end{aligned} \tag{7}$$

By using the method of Green's function, the two-point boundary value problem (7) can be changed to a Fredholm integral equation of the first kind which relates $\delta k_n(x)$, $\delta\alpha_n(x)$, $\delta\beta_n(x)$, and $\delta\rho_n(x)$ to $\delta v_n(x, s)$ as

$$\int_0^1 \{s^2\delta\rho_n v_n + s\delta\alpha_n v_n - \delta\beta_n v_n - \partial(\delta k_n\, \partial v_n/\partial x')/\partial x'\}\, G_n(x, x', s)\, dx' = \delta v_n(x, s), \quad (8)$$

where $G_n(x, x', s)$ is the Green's function of the differential operator (7) and it can be computed numerically in general.

After replacing the v_{n+1} at the right-hand side of (8) by v for the purpose of accelerating the rate of convergence and a simple integration by parts, one obtains the integral relation,

$$\int_0^1 \left\{G_n(x, x', s) v_n \cdot (s^2\delta\rho_n + s\delta\alpha_n - \delta\beta_n) + \frac{\partial G_n(x, x', s)}{\partial x'} \frac{\partial v_n}{\partial x'} \delta k_n\right\} dx'$$
$$= v(x, s) - v_n(x, s). \quad (9)$$

Upon differentiating (9) with respect to x once and setting $x = 0$, 1 and setting $x = x_1$ and x_2 in (9) one obtains, respectively, a system of four Fredholm integral equations of the first kind for the four unknowns $\delta k_n(x)$, $\delta\alpha_n(x)$, $\delta\beta_n(x)$, and $\delta\rho_n(x)$ with the help of the auxiliary data (4),

$$\int_0^1 \left\{\frac{\partial G_n(0, x', s)}{\partial x} v_n \cdot (s^2\delta\rho_n + s\delta\alpha_n - \delta\beta_n) + \frac{\partial^2 G_n(0, x', s)}{\partial x\, \partial x'} \frac{\partial v_n}{\partial x'} \delta k_n\right\} dx'$$
$$= H_1(s) - \frac{\partial v_n(0, s)}{\partial x}, \quad (10)$$

$$\int_0^1 \left\{\frac{\partial G_n(1, x', s)}{\partial x} v_n \cdot (s^2\delta\rho_n + s\delta\alpha_n - \delta\beta_n) + \frac{\partial^2 G_n(1, x', s)}{\partial x\, \partial x'} \frac{\partial v_n}{\partial x'} \delta k_n\right\} dx'$$
$$= H_2(s) - \frac{\partial v_n(1, s)}{\partial x}, \quad (11)$$

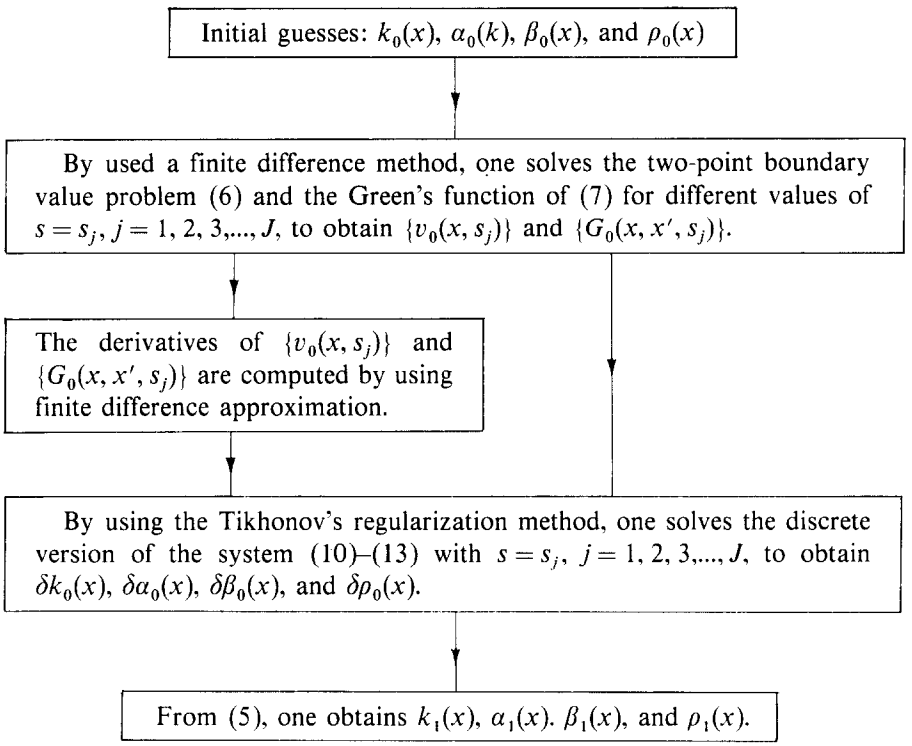
$$\int_0^1 \left\{G_n(x_1, x', s) v_n \cdot (s^2\delta\rho_n + s\delta\alpha_n - \delta\beta_n) + \frac{\partial G_n(x_1, x', s)}{\partial x'} \frac{\partial v_n}{\partial x'} \delta k_n\right\} dx'$$
$$= H_3(s) - v_n(x_1, s), \quad (12)$$

$$\int_0^1 \left\{G_n(x_2, x', s) v_n \cdot (s^2\delta\rho_n + s\delta\alpha_n - \delta\beta_n) + \frac{\partial G_n(x_2, x', s)}{\partial x'} \frac{\partial v_n}{\partial x'} \delta k_n\right\} dx'$$
$$= H_4(s) - v_n(x_2, s). \quad (13)$$

Equations (5), (6), (10)–(13) form the basic structure for each iteration in the iterative numerical algorithm of PST. Similar to the cases of single parameter inverse

problem [6–14], a numerical integration subroutine is first used to evaluate the Laplace transforms $E(s)$, $F(s)$, $H_1(s)$, $H_2(s)$, $H_3(s)$, and $H_4(s)$ at a discrete set of $s = s_j$, $j = 1, 2, 3, \ldots, J$. Then these discrete values will be used to solve Eqs. (6), (10)–(13) and the Green's function of the differential operator (7) numerically. The two-point boundary value problem (6) and Green's function of (7) can be solved by simply using a second-order finite difference method. The system of four Fredholm integral equations of the first kind can be discretized by simply using the rectangle rule; it is then solved by using the Tikhonov's regularization method [17].

The essence of the first cycle of iteration is given in the following diagram and the procedure for other cycles is exactly the same:



It is important to notice that each cycle of iteration consists basically of first solving the direct two-point boundary values problem (6) and the Green's function problem of (7) J times each and then solving the system of Fredholm integral equations of the first kind (10)–(13) once.

Numerical Simulation

In order to test the feasibility and to study the general characteristics of the PST iterative algorithm for solving the multi-parameter inverse problems of the one-

dimensional evolutional partial differential equations without the real measurement data, the following numerical simulation procedure will be carried out:

First, one chooses $k^*(x)$, $\alpha^*(x)$, $\beta^*(x)$, and $\rho^*(x)$ which are supposed to represent the correct unknown coefficients $k(x)$, $\alpha(x)$, $\beta(x)$, and $\rho(x)$, respectively. One also chooses the boundary functions $e(t)$ and $f(t)$ which are supposed to represent a part of measured data and then their Laplace transforms $E(s)$ and $F(s)$ are numericaly computed for a chosen discrete set of $s = s_j$, $j = 1, 2, 3, \ldots, J$. Next, the boundary value problem (3) is solved for each s_j, $j = 1, 2, 3, \ldots, J$, by using a simple finite difference method, and then the supposedly measured auxiliary data $H_i(s_j)$, $i = 1, 2, 3, 4$, $j = 1, 2, 3, \ldots, J$, can be generated by using simple finite difference approximations. To start the iterative algorithm, $k_0(x)$, $\alpha_0(x)$, $\beta_0(x)$, and $\rho_0(x)$ are assumed. Upon solving Eqs. (5), (6), (10)–(13) numerically, one obtains $k_1(x)$, $\alpha_1(x)$, $\beta_1(x)$, and $\rho_1(x)$. Then in a similar manner $k_2(x)$, $\alpha_2(x)$, $\beta_2(x)$, and $\rho_2(x)$ can be obtained. One continues this procedure until finally an acceptable numerical limits $k_N(x)$, $\alpha_N(x)$, $\beta_N(x)$, and $\rho_N(x)$ are reached. Other than the truncation, round-off, numerical integration, and finite difference approximation errors in both generating the numerical data and computing $k_N(x)$, $\alpha_N(x)$, $\beta_N(x)$, and $\rho_N(x)$, the L_2 multi-function norm $I_{\Sigma N} = \{I_{kN}^2 + I_{\alpha N}^2 + I_{\beta N}^2 + I_{\rho N}^2\}^{1/2}$, where $I_{\phi n}^2 = \int_0^1 |\phi^* - \phi_n|^2 \, dx$, can be used as a criterion for evaluating the performance of the iterative algorithm of the PST.

For economical reasons, the numerical simulation here is carried out only for the case of two-parameter inverse problems of the one-dimensional evolutional partial differential equations, i.e., the unknowns are either $k(x)$ and $\alpha(x)$ (the case of diffusion equation) of $k(x)$ and $\rho(x)$ (the case of wave equation). Since our past and present experiences have shown that the PST can solve the inverse problems of the diffusion equation and the wave equation with equal proficiency, it suffices to solve just one of them. Here a large class of $k^*(x)$, $k_0(x)$, $\rho^*(x)$, and $\rho_0(x)$ are used in the numerical simulation. The closed interval $0 \leqslant x \leqslant 1$ is divided into fourteen equal subintervals and $s_j = j$, $j = 1, 2, 3, \ldots, 9$, are chosen in our discretization. Moreover, a square pulse is chosen for $e(t)$ and $f(t)$ is set to zero for simplicity.

The numerical results are plotted in Figs. 1–12. The maximum norms of $k^*(x) - k_0(x)$, $k^*(x) - k_N(x)$, $\rho^*(x) - \rho_0(x)$, and $\rho^*(x) - \rho_N(x)$ for all cases can be estimated

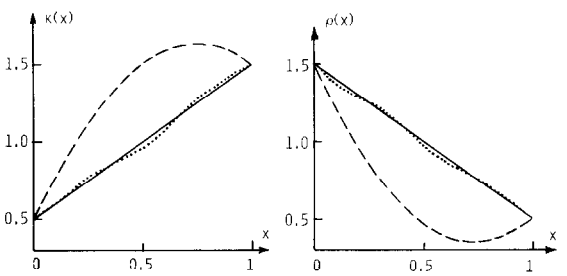


FIG. 1. Comparison of the calculated $k_9(x)$ and $\rho_9(x)(\cdots)$ and the exact $k^*(x)$ and $\rho^*(x)$ (—), with the initial guesses $k_0(x)$ and $\rho_0(x)$ (- - -).

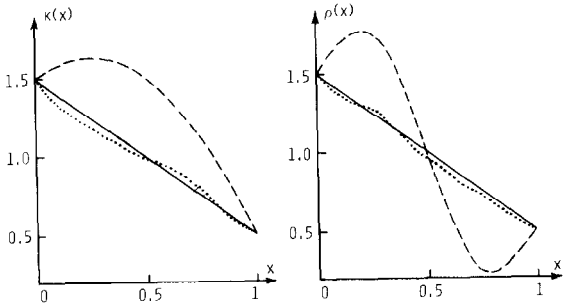


FIG. 2. Same description Fig. 1 except $N = 14$.

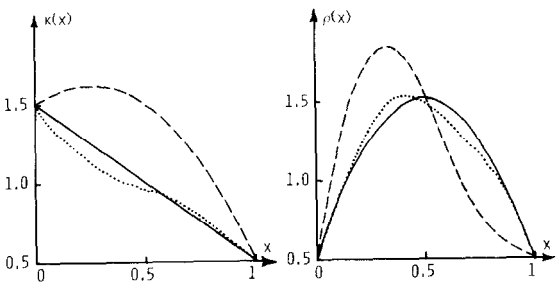


FIG. 3. Same description Fig. 1 except $N = 12$.

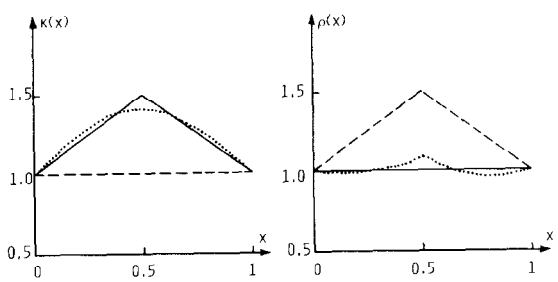


FIG. 4. Same description Fig. 1 except $N = 4$.

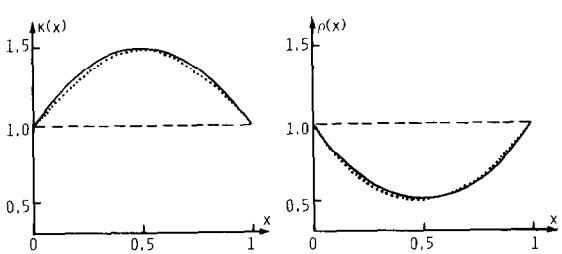


FIG. 5. Same description Fig. 1 except $N = 5$.

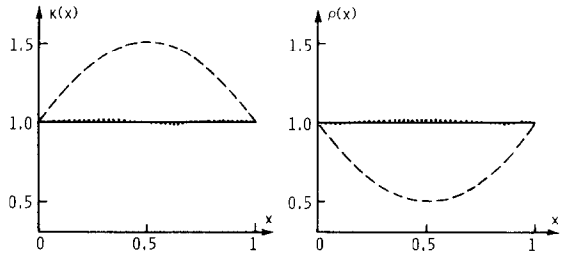


FIG. 6. Same description Fig. 1 except $N = 9$.

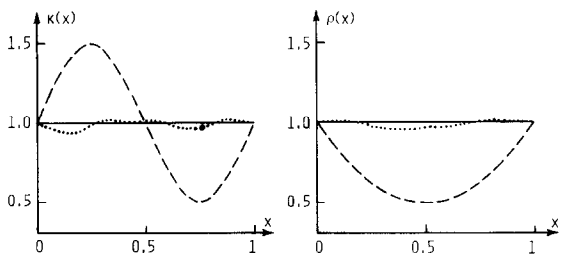


FIG. 7. Same description Fig. 1 except $N = 15$.

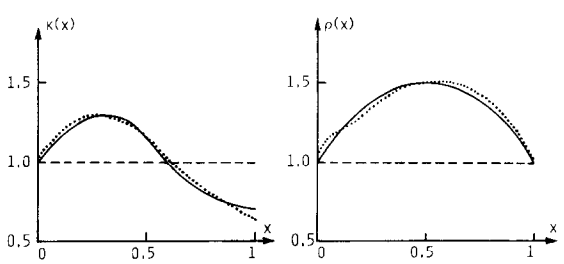


FIG. 8. Same description Fig. 1 except $N = 15$.

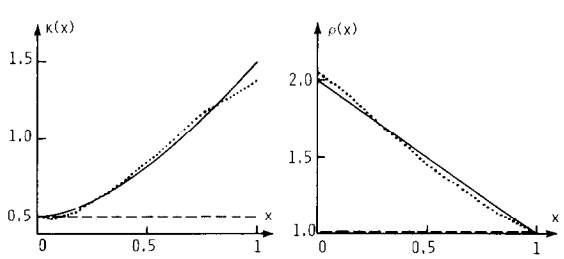


FIG. 9. Same description Fig. 1 except $N = 10$.

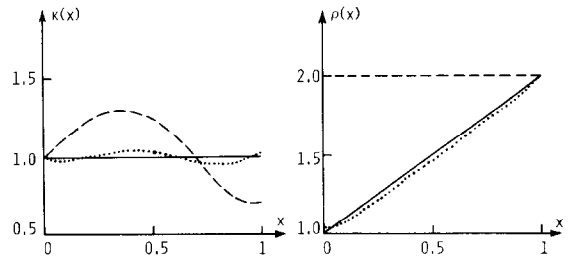


FIG. 10. Same description Fig. 1 except $N = 16$.

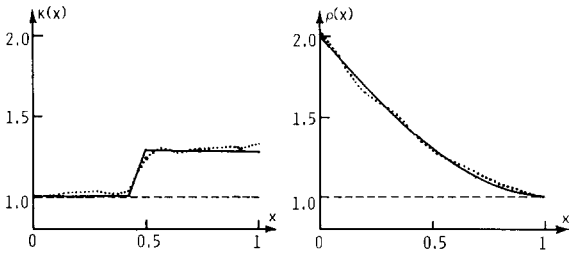


FIG. 11. Same description Fig. 1 except $N = 3$.

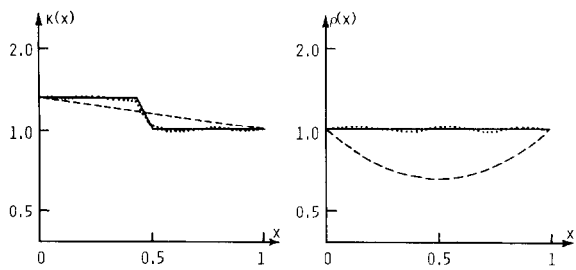


FIG. 12. Same description Fig. 1 except $N = 12$.

from the graphs in these figures. The iterative procedure is set to stop as soon as $|I_{\Sigma n+1} - I_{\Sigma n}| < 0.01$ and then $(n+1) = N$, because it will make very little difference in plotting the numerical results. As a typical example, plots of $I_{\Sigma n}$, I_{kn}, and $I_{\rho n}$ as functions of n, for example, in Fig. 10 are given in Fig. 13. The L_2 norms $I_{\Sigma 0}$, $I_{\Sigma N}$, I_{k0}, I_{kN}, $I_{\rho 0}$, and $I_{\rho N}$ for all cases are tabulated in Table I.

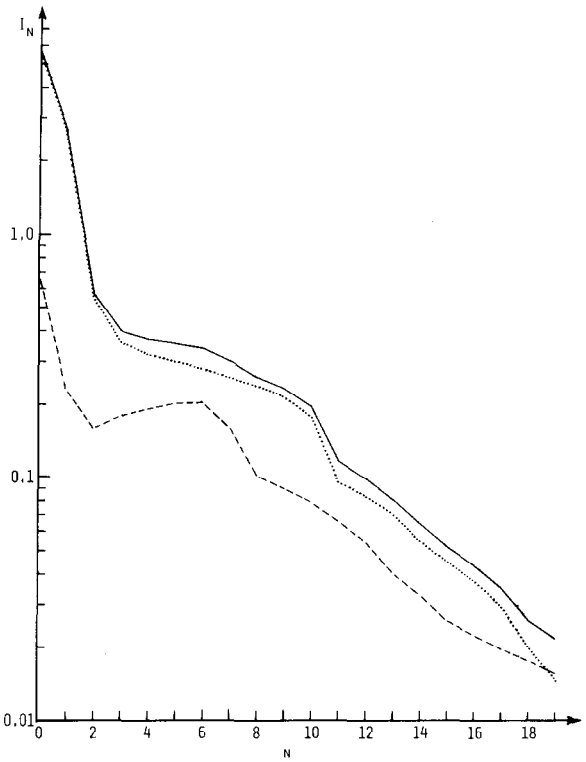


FIG. 13. I_{kn}, and $I_{\rho n}$ as functions of n for the example in Fig. 10 are shown. $I_{\Sigma n}$ (—), I_{kn} (- - -), and $I_{\rho n}$(···).

TABLE I

Fig. No.		N	I_{k0}	I_{kN}	$I_{\rho N}$	$I_{\Sigma 0}$	$I_{\Sigma N}$
1	9	1.395	0.058	1.395	0.045	1.972	0.073
2	14	1.395	0.151	1.323	0.109	1.922	0.187
3	12	1.395	0.243	1.323	0.313	1.922	0.396
4	4	1.086	0.158	1.086	0.150	1.536	0.217
5	5	1.399	0.099	1.339	0.060	1.894	0.116
6	9	0.986	0.015	0.986	0.006	1.394	0.017
7	15	0.935	0.349	0.986	0.115	1.359	0.368
8	15	0.680	0.023	2.020	0.026	2.131	0.035
9	10	3.520	0.062	6.330	0.094	7.243	0.113
10	16	0.670	0.023	5.510	0.038	5.541	0.044
11	5	0.810	0.034	3.520	0.010	3.612	0.035
12	10	0.217	0.018	0.697	0.038	0.730	0.042

DISCUSSION

Although only a small number of computational zones (both in x and s) are used in the numerical simulation here, the numerical results in Figs. 1–12 have demonstrated that the PST iterative numerical algorithm does give good results in solving the multi-parameter inverse problems of a one-dimensional linear evolutional partial differential equation, and in particular, every jump in the unknown parameters can be successively approximated within four spatial grid points. It is clear that the PST for solving the multi-parameter inverse problems is as robust as it is for solving the single parameter inverse problems [6–14]. However, for the multi-parameter inverse problems the convergence of the first few iterates of individual unknown parameter is not necessarily monotonic in the L_2 single function norm whereas it is monotonic for the single parameter inverse problems, nevertheless here the convergence is monotonic in the L_2 multi-function norm (Fig. 13).

The accuracy of the numerical algorithm can be improved greatly if a larger number of computational zones in x and s are used; more efforts are made in computing each individual step and in discretization of the partial differential equation and the integral equations in the numerical algorithm; and the values of s_j are properly chosen in solving the Fredholm integral equations of the first kind. Of course, it is counterproductive for one to increase overwhelmingly the number of computational zones in x and s, for it is well known that this will makes the discretized version of the Fredholm integral equation of the first kind more ill-conditioned.

To be sure, the PST iterative algorithm is not a method for settling the question of the uniqueness of the solution of an inverse problem. The general proof of uniqueness of the solution of the inverse problem, done quite differently, will be presented elsewhere [16]. The approximate solution obtained by using the PST is unique in the sense of being the closest one to the initial guess in the L_2 norm. Moreover, it is clear from our numerical simulation that for different initial guesses the iterations converge to slightly different numerical limits. However, if this numerical algorithm is reasonably robust, and the PST indeed is, then any one of the approximate solutions will be an acceptable approximation. This numerical computation phenomenon can be attributed to the accumulation of those nonnegligible errors in each iterate where the ill-posed Fredholm integral equation of the first kind is solved.

It is clear that the PST again fares very well in regard to the four practical criteria for the evaluation of any numerical method (stated in the Introduction) for solving the multi-parameter inverse problems of the evolutional partial differential equations. The only difference here is that more measurement data on the boundary or the interior are needed for solving the multi-parameter inverse problems. Efforts in extending the PST iterative numerical algorithm to solve multi-parameter inverse problems of a system of two- and three-dimensional coupled wave equations (elastic wave equation) are well under way and their results will be reported in the near future.

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